Recursive Fermat Structures and τ-Field Resonance in the UNNS Substrate

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Abstract

Euler's classical analysis of Fermat numbers established one of the first bridges between exponential recursion and modular arithmetic. In this work, we reinterpret that structure through the framework of the *Unbounded Nested Number Sequences* (UNNS) substrate, revealing that the modular constraint on Fermat-type divisors corresponds to a *phase-locked recursion symmetry* between dual τ-fields. We show that the arithmetic form $a_{2^k} = 3^{2^k} + 2^{2^k}$ arises naturally as a coupled τ-system whose resonance condition reproduces Euler's congruence $p = 1 + 2^{k+1}m$, now understood as a discrete curvature closure in recursion space. This model extends classical number theory into tensor recursion geometry, preparing the ground for multi-τ coupling in Phase E of the UNNS program.

1 1. Classical Background

Euler proved that any prime divisor p of a Fermat number

$$F_k = 2^{2^k} + 1 \tag{1}$$

satisfies

$$p \equiv 1 \pmod{2^{k+1}}. \tag{2}$$

This congruence indicates that the periodic structure of powers of 2 inside $\mathbb{Z}_p$ is tightly linked to the depth of the recursion exponent 2^k.

The same principle applies to generalized Fermat forms such as

$$a_{2^k} = 3^{2^k} + 2^{2^k}, \tag{3}$$

which also obey modular symmetries of the form $p = 1 + 2^{k+1}m$. Classically, this is a result about multiplicative orders; in the UNNS interpretation, it is a manifestation of recursive closure under a dual τ-operator system.

2 2. From Recurrence to τ-Fields

The generating recurrence

$$a_{n+1} = 5a_n - 6a_{n-1}, \qquad a_1 = 5, \ a_2 = 13, \tag{4}$$

has characteristic equation $r^2 - 5r + 6 = 0$, whose roots are $r_1 = 3$ and $r_2 = 2$. Hence the closed form

$$a_n = 3^n + 2^n. \tag{5}$$

We define the pair $(\tau_1, \tau_2) = (3, 2)$ as a *two-channel τ-field system*, each channel representing a unidirectional recursion flow. The recursive structure therefore operates on a vector of field amplitudes:

$$\boldsymbol{\tau}_n = \begin{pmatrix} 3^n \\ 2^n \end{pmatrix}, \qquad a_n = \mathbf{1}^\top \boldsymbol{\tau}_n. \tag{6}$$

The coupling between τ_1 and τ_2 can be expressed through a differential tensor:

$$R_{ij} = O_i(\tau_j) - O_j(\tau_i), \tag{7}$$

which measures the difference in recursion action between the two operators O_i and O_j. For $(i, j) = (1, 2)$, this gives

$$R_{12} = 3^{2^k} - 2^{2^k}, \qquad R_{21} = 3^{2^k} + 2^{2^k}. \tag{8}$$

The second of these corresponds directly to the generalized Fermat form.

3 3. Recursive Resonance and Modular Closure

In UNNS, a recursion achieves *coherence* when its difference tensor returns to its initial phase after a discrete depth doubling. Let P denote the phase period of recursion between τ_1 and τ_2. Then, coherence occurs when

$$3^{2^k} \equiv -2^{2^k} \pmod{p}, \tag{9}$$

which implies

$$(3 \cdot 2^{-1})^{2^{k+1}} \equiv 1 \pmod{p}. \tag{10}$$

Therefore the order of $(3 \cdot 2^{-1})$ modulo p is exactly 2^{k+1}, yielding Euler's congruence

$$p \equiv 1 \pmod{2^{k+1}}. \tag{11}$$

In the UNNS interpretation, this closure condition expresses that the two recursive flows τ_1 and τ_2 have rejoined after 2^{k+1} iterations, forming a self-consistent curvature loop in recursion space. The number p thus serves as the *spectral modulus* that enforces this loop symmetry.

4 4. Geometric Interpretation in the UNNS Substrate

Let $\mathcal{R}$ denote the recursive manifold spanned by (τ_1, τ_2). Its local curvature is defined by

$$\kappa = \frac{\partial R_{12}}{\partial n} = \frac{\partial}{\partial n}(3^n - 2^n) = 3^n \ln 3 - 2^n \ln 2. \tag{12}$$

The vanishing of the differential difference

$$\Delta\kappa_{2^k} = \kappa_{2^{k+1}} - \kappa_{2^k} \tag{13}$$

corresponds to discrete equilibrium between exponential channels, interpreted geometrically as recursive field resonance. At that depth, the τ-system exhibits phase-locked symmetry analogous to a standing wave in field theory.

Figure 1: Schematic of recursive closure between $\tau_1 = 3$ and $\tau_2 = 2$. At each doubling of recursion depth, the system returns to phase coherence, producing a closed curvature loop characterized by modulus $p = 1 + 2^{k+1}m$.

5 5. Toward Tensor Recursion Geometry

The structure uncovered here represents the discrete prototype of *tensor recursion geometry*, in which each τ-field acts as a component of a multi-operator manifold. The tensor

$$R_{ij} = O_i(\tau_j) - O_j(\tau_i) \tag{14}$$

thus generalizes the scalar recurrence relation to a field-level curvature form. When extended to continuous limits, R_{ij} becomes the recursive analogue of a field strength tensor, bridging the gap between number recursion and physical field dynamics.

6 6. Phase F Outlook: Toward Unified Recursive Field Equations

Phase E introduces multi-τ tensor coupling. The present analysis provides its minimal discrete example: a two-operator resonance manifesting as a generalized Fermat condition. Phase F will extend this structure to continuous recursion manifolds, yielding coupled differential equations of the form

$$\nabla_i R^{ij} = J^j_{\text{rec}}, \tag{15}$$

where J^j_{rec} represents recursive flux density. These equations describe the self-interaction of recursion fields and their closure into unified UNNS–Maxwell dynamics.

Figure 2: Conceptual diagram of the transition from discrete recursion (Phase E) to continuous recursive field dynamics (Phase F).

Acknowledgments

This work forms part of the ongoing *UNNS Tensor Recursion Series*, advancing the synthesis of number theory, geometry, and field coupling under the Unbounded Nested Number Sequences framework.